A general construction of barycentric coordinates over convex polygons

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Dedicated to Charles A. Micchelli on his 60th Birthday.

Abstract

Barycentric coordinates are unique for triangles, but there are many possible generalizations to convex polygons. In this paper we derive sharp upper and lower bounds on *all* barycentric coordinates over convex polygons and use them to show that all such coordinates have the same continuous extension to the boundary. We then present a general approach for constructing such coordinates and use it to show that the Wachspress, mean value, and discrete harmonic coordinates all belong to a unifying one-parameter family of smooth *three-point coordinates*. We show that the only members of this family that are positive, and therefore barycentric, are the Wachspress and mean value ones. However, our general approach allows us to construct several sets of smooth *five-point coordinates*, which are positive and therefore barycentric.

1 Introduction

There are many applications of triangular barycentric coordinates. They are the starting point for the representation of Bernstein-Bézier polynomials over triangles, leading to both parametric triangular Bézier patches in computer-aided design and to piecewise polynomials (splines) over triangulations, used in terrain modelling and the finite element method. Barycentric coordinates are also useful for simply representing a point in a triangle as a convex combination of the vertices, and frequently occur in computer graphics when modelling geometry with triangular meshes.

In many of these applications, it would be useful to generalize barycentric coordinates to convex polygons and this is the topic of this paper. Let Ω be a convex polygon in the plane, regarded as a closed set, with vertices $v_1, v_2, \ldots, v_n$, $n \geq 3$, in an anticlockwise ordering, see Figure 1. We will call any set of functions $\lambda_i : \Omega \to \mathbb{R}$, $i = 1, \ldots, n$, *barycentric coordinates* if they satisfy, for all $v \in \Omega$, the three properties

$$\lambda_i(v) \geq 0, \qquad i = 1, 2, \ldots, n, \tag{1}$$

$$\sum_{i=1}^{n} \lambda_i(v) = 1 \tag{2}$$

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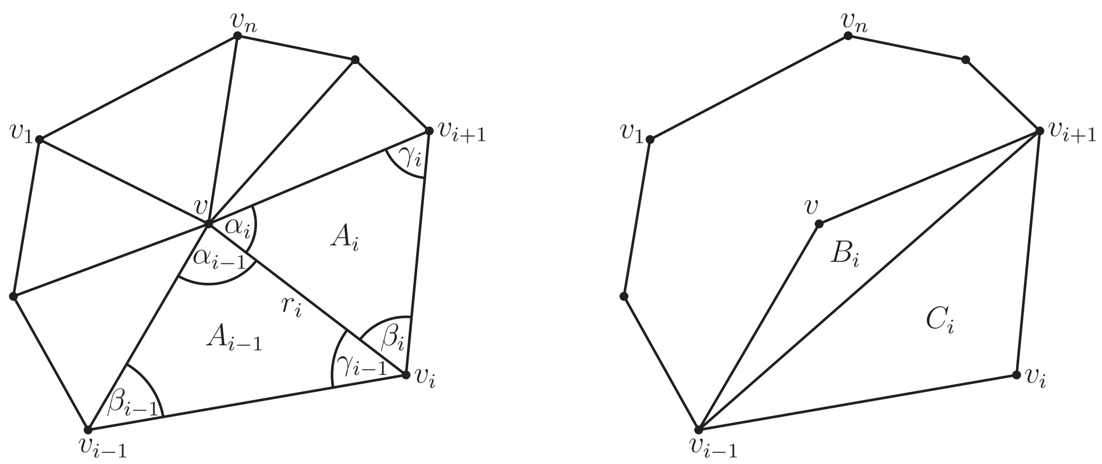

Figure 1: Notation for convex polygons.

and

$$\sum_{i=1}^{n} \lambda_i(v) v_i = v. \tag{3}$$

This definition does indeed generalize triangular barycentric coordinates, for in the case $n = 3$, Ω is the triangle $[v_1, v_2, v_3]$, and the properties (2) and (3) alone determine the three coordinates uniquely, namely

$$\lambda_1(v) = \frac{A(v, v_2, v_3)}{A(v_1, v_2, v_3)}, \quad \lambda_2(v) = \frac{A(v_1, v, v_3)}{A(v_1, v_2, v_3)}, \quad \lambda_3(v) = \frac{A(v_1, v_2, v)}{A(v_1, v_2, v_3)},$$

where $A(u_1, u_2, u_3)$ denotes the signed area of the triangle $[u_1, u_2, u_3]$. For example, with $v = (x, y)$ and $v_i = (x_i, y_i)$, we have

$$A(v, v_2, v_3) = \frac{1}{2} \begin{vmatrix} 1 & 1 & 1 \\ x & x_2 & x_3 \\ y & y_2 & y_3 \end{vmatrix}.$$

The functions $\lambda_1, \lambda_2, \lambda_3 : \Omega \to \mathbb{R}$ are thus linear polynomials and non-negative and have the Lagrange property, $\lambda_i(v_j) = \delta_{ij}$.

Until very recently, much if not all work on generalizations to arbitrary n was restricted to what are often now called the *Wachspress coordinates* [14],

$$\lambda_i(v) = \frac{w_i(v)}{\sum_{j=1}^{n} w_j(v)}, \tag{4}$$

with

$$w_i(v) = A(v_{i-1}, v_i, v_{i+1}) \prod_{j \neq i-1, i} A(v, v_j, v_{j+1}). \tag{5}$$

While Wachspress's application was finite elements methods, Charrot and Gregory [2] and Loop and DeRose [9] wanted coordinates for building n-sided Bézier patches for computer-aided design (see also Goldman [8]). Independently of Wachspress's work, Loop and DeRose [9] proposed the coordinates λ_i of (4) but with (5) replaced by

$$w_i(v) = \prod_{j \neq i-1,i} A(v, v_j, v_{j+1}), \tag{6}$$

and showed that they are barycentric (i.e. they satisfy the linear precision property (3)) in the case that Ω is a *regular* polygon. We note that the coordinates defined by (6) are equivalent to Wachspress's when Ω is a regular polygon because the areas $A(v_{i-1}, v_i, v_{i+1})$ in that case are equal. Charrot and Gregory [2] used the same coordinates with Ω a regular pentagon ($n = 5$).

Later, Warren [15] brought Wachspress's work to light and also generalized some of Wachspress's constructions to higher dimensions. However, we first learned about Wachspress's coordinates through the paper by Meyer, Lee, Barr, and Desbrun [10], who independently proved that these coordinates have the linear precision property (3) using the equivalent local form,

$$w_i(v) = \frac{A(v_{i-1}, v_i, v_{i+1})}{A(v, v_{i-1}, v_i) A(v, v_i, v_{i+1})}. \tag{7}$$

Recently, Warren, Schaefer, Hirani, and Desbrun [17] have generalized this latter approach to convex polyhedra in arbitrary dimensions.

Wachspress coordinates have many nice properties, such as being affine invariant; they are unchanged when any affine transformation $T : \mathbb{R}^2 \to \mathbb{R}^2$ is applied simultaneously to the points $v_1, \ldots, v_n$ and v. Specifically, if we express λ_i as $\lambda_i(v; v_1, \ldots, v_n)$ to indicate the dependency on the vertices of Ω, we have

$$\lambda_i(Tv; Tv_1, \ldots, Tv_n) = \lambda_i(v; v_1, \ldots, v_n). \tag{8}$$

Wachspress coordinates are clearly smooth (C^∞) and also rational polynomials in the coordinates x and y of the point $v = (x, y)$ with degree at most $n - 2$. Warren [16] has shown that this is the least possible degree.

Then recently some new, smooth barycentric coordinates were constructed in [6]. With λ_i as in (4), these so-called *mean value coordinates* are defined by

$$w_i(v) = \frac{\tan(\alpha_i(v)/2) + \tan(\alpha_{i-1}(v)/2)}{\|v_i - v\|}. \tag{9}$$

Here, $\alpha_i(v)$ denotes the angle in the triangle $[v, v_i, v_{i+1}]$ at the vertex v; see Figure 1. Though these coordinates are not affine invariant, they have the invariance (8) whenever T is a similarity, i.e. a combination of translation, rotation, or uniform scaling. The motivation and advantage of these coordinates is that they allow a vertex of a planar triangulation to be expressed as a convex combination of its neighbouring vertices, i.e., they allow Ω to be non-convex, specifically star-shaped, provided v belongs to the kernel of Ω. Mean value coordinates have been applied to surface parameterization in [6] and to morphing of triangulations and polygons in [13]. Of course, under our assumption that Ω is convex, the kernel of Ω is the whole of its interior $\mathrm{Int}(\Omega)$, so the mean value coordinates are well defined for all v in $\mathrm{Int}(\Omega)$, but, as we will see, these coordinates extend in a natural way to the boundary $\partial\Omega$ as well.

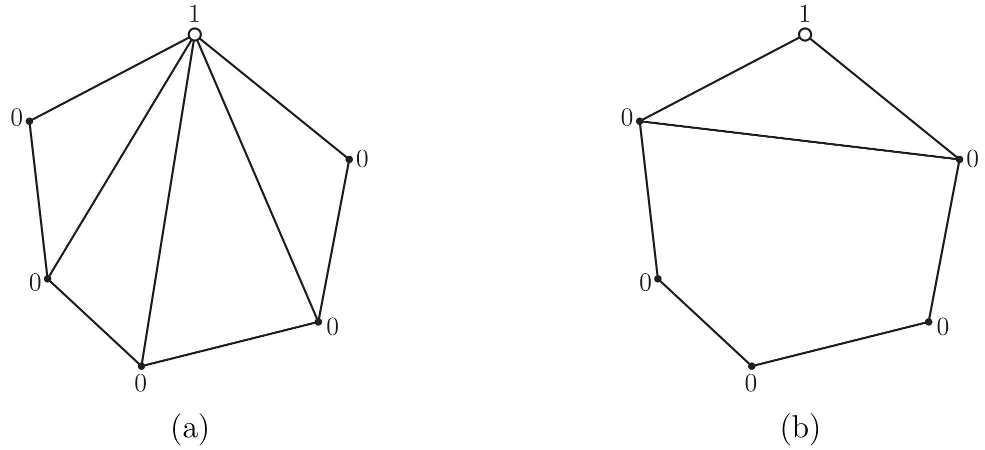

Figure 2: Upper bound (a) and lower bound (b).

The fact that we now had two different sets of smooth barycentric coordinates for convex polygons prompted us to investigate barycentric coordinates in more generality: both to explore their general properties and to try to construct more coordinates with desirable properties.

We begin in Section 2 by deriving sharp upper and lower bounds on each coordinate function λ_i and use them to show that *all* barycentric coordinates which are continuous in the interior $\mathrm{Int}(\Omega)$, such as in (9), extend *continuously* to the boundary $\partial\Omega$.

In Section 3 we give a general method for constructing barycentric coordinates. Then specializing to *3-point coordinates*, we show in Section 4 that the Wachspress, mean value, and *discrete harmonic coordinates* [11, 4] are all related in the sense that they all belong to an interesting one-parameter family of functions. We show, however, that the *only* members of this family which are true barycentric coordinates (i.e. positive) are the Wachspress and mean value ones. This motivates us to widen our net and to consider *5-point coordinates* in Section 5, and we succeed in constructing several new sets of smooth barycentric coordinates. Some, like the Wachspress coordinates, have affine invariance and others, like the mean value coordinates, extend to the kernels of star-shaped polygons. We complete the paper with some illustrative numerical examples.

2 General properties

An important and easily derived property which Wachspress's coordinates inherit from barycentric coordinates over triangles is that they are linear along each edge $[v_j, v_{j+1}]$ of Ω and satisfy the Lagrange property $\lambda_i(v_j) = \delta_{ij}$ at the vertices. In what follows we will show that all barycentric coordinates which are continuous in $\mathrm{Int}(\Omega)$ extend *continuously* to these same boundary values. We achieve this by deriving upper and lower bounds on each coordinate function λ_i over Ω, which agree on the boundary $\partial\Omega$. We also show that these two bounds are sharp and we think this makes them interesting in their own right.

For each $i \in \{1, \ldots, n\}$, let $L_i : \Omega \to \mathrm{I\!R}$ be the continuous piecewise linear function which is linear on each triangle of the form $[v_j, v_{j+1}, v_i]$, $j \neq i-1, i$, and has the values $L_i(v_j) = \delta_{ij}$ at the vertices of Ω; as in Figure 2a.

Conversely, let $\ell_i : \Omega \to \mathrm{I\!R}$ be the function which is linear on the triangle $[v_{i-1}, v_i, v_{i+1}]$

and zero on $\Omega \setminus [v_{i-1}, v_i, v_{i+1}]$, with $\ell_i(v_j) = \delta_{ij}$ at the vertices; see Figure 2b.

Proposition 1 *Let D be any subset of Ω and suppose the functions $\lambda_1, \ldots, \lambda_n : D \to \mathbb{R}$ satisfy the three defining properties of barycentric coordinates, (1), (2) and (3), for $v \in D$. Then, for $i = 1, \ldots, n$,*

$$0 \le \ell_i(v) \le \lambda_i(v) \le L_i(v) \le 1, \qquad v \in D. \tag{10}$$

Proof. Any point $v \in D$, being a point in Ω, belongs to at least one triangle $[v_i, v_j, v_{j+1}]$, $j \neq i-1, i$. The two properties (2) and (3) imply linear precision and since $A(v, v_j, v_{j+1})$ is linear in v,

$$A(v, v_j, v_{j+1}) = \sum_{k=1}^{n} \lambda_k(v) A(v_k, v_j, v_{j+1}) \ge \lambda_i(v) A(v_i, v_j, v_{j+1}),$$

the latter inequality due to the positivity property (1). This implies that

$$\lambda_i(v) \le A(v, v_j, v_{j+1}) / A(v_i, v_j, v_{j+1}) = L_i(v).$$

The opposite inequality, $\ell_i(v) \le \lambda_i(v)$, is trivial if v is outside the triangle $[v_{i-1}, v_i, v_{i+1}]$, for then $\ell_i(v) = 0$. So suppose $v \in [v_{i-1}, v_i, v_{i+1}]$. Then, since $A(v, v_{i+1}, v_{i-1})$ is linear in v,

$$\begin{aligned} A(v, v_{i+1}, v_{i-1}) &= \sum_{k=1}^{n} \lambda_k(v) A(v_k, v_{i+1}, v_{i-1}) \\ &= \lambda_i(v) A(v_i, v_{i+1}, v_{i-1}) - \sum_{k \neq i-1, i, i+1} \lambda_k(v) A(v_k, v_{i-1}, v_{i+1}) \\ &\le \lambda_i(v) A(v_i, v_{i+1}, v_{i-1}) \end{aligned}$$

which implies that

$$\lambda_i(v) \ge A(v, v_{i+1}, v_{i-1}) / A(v_i, v_{i+1}, v_{i-1}) = \ell_i(v).$$

□

Letting $D = \Omega$, and noting that L_i and ℓ_i agree on the boundary $\partial\Omega$, we have the following consequence.

Corollary 1 *If $\lambda_1, \ldots, \lambda_n : \Omega \to \mathbb{R}$ is any set of barycentric coordinates then, for $i = 1, \ldots, n$,*

$$\ell_i(v) \le \lambda_i(v) \le L_i(v), \qquad v \in \Omega. \tag{11}$$

and

$$\ell_i(v) = \lambda_i(v) = L_i(v), \qquad v \in \partial\Omega, \tag{12}$$

and therefore λ_i is linear on each edge $[v_j, v_{j+1}]$ and satisfies $\lambda_i(v_j) = \delta_{ij}$.

A further consequence is that it is sufficient to define barycentric coordinates only in the interior of Ω.

Corollary 2 *Suppose $\lambda_1, \ldots, \lambda_n : \mathrm{Int}(\Omega) \to \mathbb{R}$ are continuous and satisfy the three properties (1), (2) and (3) on $\mathrm{Int}(\Omega)$. Then they are continuous barycentric coordinates in the sense that they have a unique continuous extension to Ω and therefore satisfy (1), (2) and (3) on the whole of Ω.*

Proof. If w is any point in $\partial\Omega$ and $w_1, w_2, \ldots$, is any sequence of points in $\mathrm{Int}(\Omega)$ such that $\lim_{k\to\infty} w_k = w$, then since

$$\ell_i(w_k) \le \lambda_i(w_k) \le L_i(w_k),$$

and

$$\lim_{k\to\infty} \ell_i(w_k) = \lim_{k\to\infty} L_i(w_k) = \ell_i(w),$$

we have

$$\lim_{k\to\infty} \lambda_i(w_k) = \ell_i(w) = L_i(w).$$

□

Thus from now on, if we construct continuous functions $\lambda_1, \ldots, \lambda_n$ satisfying the three basic properties (1), (2) and (3) in $\mathrm{Int}(\Omega)$, we are justified in viewing them as continuous barycentric coordinates on the whole of Ω.

The reader might now be wondering whether the functions $L_1, \ldots, L_n$ themselves form a set of barycentric coordinates and the same for $\ell_1, \ldots, \ell_n$. However, a little thought shows that this is not the case for $n > 3$. For example with $n = 4$, Ω a square, and v its centre, we have $L_i(v) = 1/2$ and $\ell_i(v) = 0$ for all $i = 1, 2, 3, 4$ and so $\sum_i L_i(v) = 2 > 1$ and $\sum_i \ell_i(v) = 0 < 1$.

Nonetheless, for *each* $i = 1, \ldots, n$, the functions L_i and ℓ_i are special because, as we will show next, they are themselves i-th barycentric coordinates. We will call any function $\lambda_i : \Omega \to \mathbb{R}$ an *i-th barycentric coordinate* if it is the i-th member of *some* set of barycentric coordinates $\lambda_1, \ldots, \lambda_n : \Omega \to \mathbb{R}$.

This immediately implies that L_i and ℓ_i are the *least* upper bound and *greatest* lower bound on λ_i in inequality (11).

Proposition 2 *The functions L_i and ℓ_i are i-th barycentric coordinates $\Omega \to \mathbb{R}$.*

Proof. To see that L_i is an i-th barycentric coordinate, let $\mathcal{T}_i$ be the triangulation consisting of the $n-2$ triangles $[v_j, v_{j+1}, v_i]$, $j \ne i-1, i$. Then let $L_{i,1}, \ldots, L_{i,n} : \Omega \to \mathbb{R}$ be the continuous piecewise linear functions which are linear on each triangle in $\mathcal{T}_i$ and satisfy $L_{i,k}(v_j) = \delta_{kj}$. Then $L_{i,1}, \ldots, L_{i,n}$ are clearly a set of barycentric coordinates, because in each triangle $[v_j, v_{j+1}, v_i]$, $j \ne i-1, i$, only $L_{i,j}$, $L_{i,j+1}$, and $L_{i,i}$ are non-zero and these three functions are themselves barycentric coordinates with respect to $[v_j, v_{j+1}, v_i]$. Thus $L_i = L_{i,i}$ is an i-th barycentric coordinate.

Conversely, let $\mathcal{S}_i$ be *any* triangulation of Ω in which one of the triangles is $[v_{i-1}, v_i, v_{i+1}]$, and let $\ell_{i,1}, \ldots, \ell_{i,n} : \Omega \to \mathbb{R}$ be the associated continuous piecewise linear functions, i.e., which are linear on each triangle in $\mathcal{S}_i$ and such that $\ell_{i,k}(v_j) = \delta_{kj}$. Then $\ell_{i,1}, \ldots, \ell_{i,n}$ are also a set of barycentric coordinates and so $\ell_i = \ell_{i,i}$ is therefore an i-th barycentric coordinate. □

In fact all functions between the upper and lower bounds are i-th barycentric coordinates.

Proposition 3 *A function $\lambda_i : \Omega \to \mathbb{R}$ is an i-th barycentric coordinate if and only if*

$$\ell_i(v) \le \lambda_i(v) \le L_i(v), \qquad v \in \Omega. \tag{13}$$

Proof. We have already shown that inequality (13) is necessary for λ_i to be an i-th barycentric coordinate. To show that the inequality is also sufficient, note that it implies that

$$\lambda_i = (1-\alpha)\ell_i + \alpha L_i$$

for some function $\alpha : \Omega \to \mathbb{R}$ with $0 \le \alpha \le 1$. But now, referring to the proof of Proposition 2, let

$$\lambda_{i,j} = (1-\alpha)\ell_{i,j} + \alpha L_{i,j}$$

for $j = 1, \ldots, n$. It is easy to check that $\lambda_{i,1}, \ldots, \lambda_{i,n}$ form a set of barycentric coordinates and since $\lambda_i = \lambda_{i,i}$, this means that λ_i is indeed an i-th barycentric coordinate. $\square$

The above results lead to one further interesting property of barycentric coordinates for convex polygons. Since the function ℓ_i is clearly convex as well as satisfying $\ell_i(v_j) = \delta_{ij}$, a result of Dahmen and Micchelli [3] (see also [1]) shows that ℓ_i is the pointwise maximum of all convex functions f on Ω which satisfy $f(v_j) = \delta_{ij}$. Conversely, L_i is concave and the pointwise minimum of all concave functions f on Ω such that $f(v_j) = \delta_{ij}$. Therefore

Proposition 4 *Suppose $\lambda_1, \ldots, \lambda_n : \Omega \to \mathbb{R}$ are barycentric coordinates. For each i, if λ_i is convex then $\lambda_i = \ell_i$ and if λ_i is concave then $\lambda_i = L_i$. So if λ_i is differentiable in* $\mathrm{Int}(\Omega)$ *then it is neither convex nor concave.*

This last result confirms what we have seen in all our numerical examples. We tend to see coordinate functions which are "saddle-like", i.e. at each point the Hessian is indefinite. This suggests letting λ_i be the unique solution to the Laplace equation

$$\Delta \lambda_i = 0,$$

subject to the Dirichlet boundary condition

$$\lambda_i|_{\partial\Omega} = \ell_i|_{\partial\Omega}.$$

Using the maximum principle [12] and the fact that linear bivariate functions are harmonic, it is easy to show that these *harmonic coordinates* $\lambda_1, \ldots, \lambda_n$ constitute a set of smooth barycentric coordinates with similarity invariance. Moreover, they clearly extend to *any* polygon. However, since it appears that no explicit formula for these coordinates is known, they are less attractive for computational purposes.

We make one final remark in this section. Though we are primarily concerned with smooth coordinates, it can easily be checked that the piecewise linear functions

$$\lambda_i := \frac{1}{n} \sum_{k=1}^{n} L_{k,i},$$

where the functions $L_{i,j}$ were defined in the proof of Proposition 2, are the *shape-preserving* coordinates of [5], which are only C^0.

3 Construction of coordinates

We will now describe a general method of constructing smooth barycentric coordinates, and in particular show that there are many sets of smooth barycentric coordinates other than the Wachspress and mean value coordinates. We will only consider barycentric coordinates λ_i which are invariant to similarities. In fact Wachspress coordinates are special in that they have full affine invariance, i.e. they are in addition invariant to non-uniform scalings (e.g. a scaling in just the x coordinate). However, in many applications, uniform scale invariance is usually sufficient.

First we note that due to Corollary 2, we have only to find continuous functions λ_i satisfying (1), (2) and (3) for points v in the interior of Ω, in order to construct continuous barycentric coordinates on the whole of Ω. However, we wish to find coordinates which are as smooth as possible in the interior. Next, we note that the normalization in (4) is a common feature of many sets of barycentric coordinates. Clearly, if we can find functions $w_i : \mathrm{Int}(\Omega) \to \mathbb{R}$ such that

$$w_i(v) > 0, \qquad i = 1, 2, \ldots, n, \tag{14}$$

and

$$\sum_{i=1}^{n} w_i(v)(v_i - v) = 0, \tag{15}$$

then the normalized functions λ_i in (4) are well-defined and satisfy (1), (2) and (3). For convenience, we will call any set of functions $w_i : \mathrm{Int}(\Omega) \to \mathbb{R}$ satisfying (15) *homogeneous coordinates.* Our approach is to first give a very general formula for homogeneous coordinates. Later we will search for special cases in which the coordinates are also positive. In what follows we will define the three signed triangle areas

$$A_i(v) = A(v, v_i, v_{i+1}), \qquad B_i(v) = A(v, v_{i-1}, v_{i+1}), \qquad C_i = A(v_{i-1}, v_i, v_{i+1}). \tag{16}$$

We note that

$$A_{i-1}(v) + A_i(v) = B_i(v) + C_i, \tag{17}$$

and that due to the convexity of Ω, we have $A_i(v) > 0$ for all v in $\mathrm{Int}(\Omega)$, and $C_i > 0$, but the sign of $B_i(v)$ depends on which side of the line segment $[v_{i-1}, v_{i+1}]$ the point v is located; see Figure 1.

Proposition 5 *Let* $c_1, c_2, \ldots, c_n : \mathrm{Int}(\Omega) \to \mathbb{R}$ *be any real functions. Then the functions*

$$w_i = \frac{c_{i+1}A_{i-1} - c_i B_i + c_{i-1}A_i}{A_{i-1}A_i}, \tag{18}$$

are homogeneous coordinates, i.e., satisfy Equation (15).

Proof. By expressing $v \in \mathrm{Int}(\Omega)$ as a barycentric combination of v_{i-1}, v_i, v_{i+1}, we have

$$A_i(v)(v_{i-1} - v) + A_{i-1}(v)(v_{i+1} - v) - B_i(v)(v_i - v) = 0, \tag{19}$$

or alternatively,

$$D_i(v) := \frac{v_{i-1} - v}{A_{i-1}(v)} + \frac{v_{i+1} - v}{A_i(v)} - \frac{B_i(v)}{A_{i-1}(v)A_i(v)}(v_i - v) = 0. \tag{20}$$

Thus any linear combination of the D_i is also zero, i.e.,

$$\sum_{i=1}^{n} c_i(v) D_i(v) = 0.$$

But this equation can be rearranged, by two changes of summation index, to be Equation (15) with weights w_i given by (18). $\square$

We have now further simplified our task. No matter which functions $c_1, \ldots, c_n$ we choose, the functions $w_1, \ldots, w_n$ in (18) will be homogeneous coordinates. It remains to choose $c_1, \ldots, c_n$ to ensure that the w_i are also positive in $\mathrm{Int}(\Omega)$.

As we will see later, the point about the particular form of (18) is that if each function c_i is chosen to depend only on v_i and one or more neighbours on either side of v_i (e.g. v_{i-1} and v_{i+1}), then the function w_i will also have a local dependence. However, there is really nothing special about the form of the functions w_i in (18); they can all be expressed in this way.

Corollary 3 *Any set of homogeneous coordinates* $w_1, \ldots, w_n : \mathrm{Int}(\Omega) \to \mathbb{R}$ *can be expressed in the form of (18).*

Proof. Let c_1 and c_2 be arbitrary functions $\mathrm{Int}(\Omega) \to \mathbb{R}$ and define $c_3, \ldots, c_n$ recursively by

$$c_{i+1} = (A_{i-1} A_i w_i + c_i B_i - c_{i-1} A_i)/A_{i-1}, \qquad i = 2, \ldots, n-1. \tag{21}$$

Now set

$$\hat{w}_i = \frac{c_{i+1} A_{i-1} - c_i B_i + c_{i-1} A_i}{A_{i-1} A_i}, \qquad i = 1, \ldots, n, \tag{22}$$

and we claim that $\hat{w}_i = w_i$ for $i = 1, \ldots, n$. That this holds for $i = 2, \ldots, n-1$ follows immediately from (21). Moreover, Proposition 5 shows that the $\hat{w}_i$ satisfy (15) and so

$$\begin{aligned} \hat{w}_1(v)(v_1 - v) + \hat{w}_n(v)(v_n - v) &= -\sum_{i=2}^{n-1} \hat{w}_i(v)(v_i - v) \\ &= -\sum_{i=2}^{n-1} w_i(v)(v_i - v) = w_1(v)(v_1 - v) + w_n(v)(v_n - v). \end{aligned}$$

Since $v_1 - v$ and $v_n - v$ are linearly independent vectors, we can equate coefficients, yielding $\hat{w}_1 = w_1$ and $\hat{w}_n = w_n$. $\square$

Note that there is an alternative way of expressing w_i in (18) which may be useful at times. If we let

$$r_i(v) = \|v - v_i\|,$$

and recall that $\alpha_i(v)$ denotes the angle in the triangle $[v, v_i, v_{i+1}]$ at v, then we have

$$\begin{aligned} A_i(v) &= r_i(v) r_{i+1}(v) \sin \alpha_i(v)/2, \\ B_i(v) &= r_{i-1}(v) r_{i+1}(v) \sin(\alpha_{i-1}(v) + \alpha_i(v))/2, \end{aligned}$$

and it follows that

$$w_i = \frac{2}{r_i}\left(\frac{c_{i+1}}{r_{i+1}\sin\alpha_i} - \frac{c_i\sin(\alpha_{i-1}+\alpha_i)}{r_i\sin\alpha_{i-1}\sin\alpha_i} + \frac{c_{i-1}}{r_{i-1}\sin\alpha_{i-1}}\right) \tag{23}$$

or, letting $a_i = c_i/r_i$,

$$w_i = \frac{2}{r_i}\left(\frac{a_{i+1} - a_i\cos\alpha_i}{\sin\alpha_i} + \frac{a_{i-1} - a_i\cos\alpha_{i-1}}{\sin\alpha_{i-1}}\right). \tag{24}$$

4 Three-point coordinates

Let us begin by setting $c_i = r_i^p$ in (18), where p is any real number (positive or negative). We then obtain an interesting one-parameter family of homogeneous coordinates

$$w_{i,p} = \frac{r_{i+1}^p A_{i-1} - r_i^p B_i + r_{i-1}^p A_i}{A_{i-1}A_i}, \tag{25}$$

which are C^∞ in $\mathrm{Int}(\Omega)$, and associated normalized functions

$$\lambda_{i,p} = \frac{w_{i,p}}{\sum_{j=1}^n w_{j,p}}, \tag{26}$$

which are valid whenever the numerator is non-zero. Since each coordinate $w_{i,p}$ depends only on v_{i-1}, v_i, and v_{i+1}, we call the $w_{i,p}$ *three-point coordinates*. It is easy to see that the functions $\lambda_{i,p}$ are invariant to similarities. It may at times be helpful to use the angle formulation (24) which gives

$$w_{i,p} = \frac{2}{r_i}\left(\frac{r_{i+1}^{p-1} - r_i^{p-1}\cos\alpha_i}{\sin\alpha_i} + \frac{r_{i-1}^{p-1} - r_i^{p-1}\cos\alpha_{i-1}}{\sin\alpha_{i-1}}\right). \tag{27}$$

We next see how the Wachspress and mean value coordinates are two members of this one-parameter family.

Proposition 6 *The functions $\lambda_{i,0}$ are the Wachspress coordinates and the functions $\lambda_{i,1}$ are the mean value coordinates.*

Proof. Recalling identity (17), if we let $p = 0$ in (25), we get

$$w_{i,0} = \frac{C_i}{A_{i-1}A_i}, \tag{28}$$

which is equivalent to (7). If on the other hand we let $p = 1$, then Equation (27) implies

$$w_{i,1} = \frac{2}{r_i}\left(\frac{1-\cos\alpha_i}{\sin\alpha_i} + \frac{1-\cos\alpha_{i-1}}{\sin\alpha_{i-1}}\right), \tag{29}$$

which is equivalent to (9) up to the factor 2 that cancels out in the normalization (4). □

We have now seen two choices of the exponent p which lead to barycentric coordinates $\lambda_{i,p}$. An obvious question is whether there are other such choices. Before treating this general question, consider next a third interesting choice, namely $p = 2$. From Equation (27), we have

$$w_{i,2} = 2\left(\frac{r_{i+1} - r_i \cos\alpha_i}{r_i \sin\alpha_i} + \frac{r_{i-1} - r_i \cos\alpha_{i-1}}{r_i \sin\alpha_{i-1}}\right), \tag{30}$$

and using a little trigonometry, we find

$$w_{i,2} = 2(\cot\gamma_i + \cot\beta_{i-1}), \tag{31}$$

where β_{i-1} is the angle in the triangle $[v, v_{i-1}, v_i]$ at corner v_{i-1} and γ_i the angle in the triangle $[v, v_i, v_{i+1}]$ at corner v_{i+1}; see Figure 1. We recognize these homogeneous coordinates, once normalized, as the discrete harmonic coordinates arising from the standard piecewise linear finite element approximation to the Laplace equation studied by Pinkall and Polthier [11] and Eck et al. [4]. We have therefore shown

Proposition 7 *The functions $\lambda_{i,2}$ are the discrete harmonic coordinates.*

However, as is well known, these coordinates are rarely *barycentric* coordinates for they are rarely positive for an arbitrary convex polygon Ω. In fact, using the fact that $\cot\gamma_i + \cot\beta_{i-1}$ is positive if and only if $\gamma_i + \beta_{i-1} < \pi$ (see [7]), we establish the following:

Proposition 8 *If Ω is a convex polygon such that the discrete harmonic coordinates $\lambda_{i,2}$ are positive at all points in its interior then all its vertices $v_1, \ldots, v_n$ lie on a circle.*

Proof. It is enough to show that every sequence of four vertices v_{i-1}, v_i, v_{i+1}, v_{i+2} are co-circular. Let Q denote the quadrilateral formed by v_{i-1}, v_i, v_{i+1}, v_{i+2}. From [7] we know that $\sum_{j=1}^{n} w_{j,2}(v)$ is positive for any $v \in \mathrm{Int}(\Omega)$. Therefore, since $\lambda_{i,2}(v)$ is positive, so is $w_{i,2}(v)$ and so $\gamma_i + \beta_{i-1} < \pi$. Letting v converge to v_{i+2}, this means that the sum of the two opposite interior angles of Q at the vertices v_{i-1} and v_{i+1} is at most π. Conversely, since $\lambda_{i+1,2}(v) > 0$, letting $v \to v_{i-1}$ shows that the sum of the two other interior angles of Q is also less than or equal to π. But this is only possible if both pairs of opposite angles sum to exactly π in which case v_{i-1}, v_i, v_{i+1}, v_{i+2} are co-circular. $\square$

Somewhat curiously, it turns out that in this special circle case, the discrete harmonic coordinates are in any case *the same* as the Wachspress coordinates. We first observed this equivalence numerically but later found the following proof. As far as we know, this has not been noticed before.

Proposition 9 *If the vertices $v_1, \ldots, v_n$ lie on the unit circle around the origin, then*

$$w_{i,2}(v) = (1 - \|v\|^2) w_{i,0}(v), \qquad v \in \mathrm{Int}(\Omega), \tag{32}$$

and therefore $\lambda_{i,2} = \lambda_{i,0}$.

Proof. Fix $v \in \mathrm{Int}(\Omega)$ and let $r = \|v\|$. By rotational invariance we may assume that $v = (r, 0)$ and using polar coordinates we have

$$v_i = (\cos\varphi_i, \sin\varphi_i)$$

for some $\varphi_1, \ldots, \varphi_n$. Then

$$r_i^2 = \|v_i - v\|^2 = (\cos\varphi_i - r)^2 + (\sin\varphi_i)^2 = 1 + r^2 - 2r\cos\varphi_i. \tag{33}$$

In this particular representation we can write the triangle areas from (16) as

$$\begin{aligned} A_i &= ((\cos\varphi_i - r)\sin\varphi_{i+1} - \sin\varphi_i(\cos\varphi_{i+1} - r))/2 \\ B_i &= ((\cos\varphi_{i-1} - r)\sin\varphi_{i+1} - \sin\varphi_{i-1}(\cos\varphi_{i+1} - r))/2 \end{aligned}$$

and establish the identity

$$\cos\varphi_{i+1}A_{i-1} - \cos\varphi_i B_i + \cos\varphi_{i-1}A_i = r(A_{i-1} - B_i + A_i). \tag{34}$$

Equations (25), (33) and (34) now imply

$$\begin{aligned} w_{i,2}A_iA_{i-1} &= r_{i+1}^2 A_{i-1} - r_i^2 B_i + r_{i-1}^2 A_i \\ &= (1+r^2)(A_{i-1} - B_i + A_i) - 2r(\cos\varphi_{i+1}A_{i-1} - \cos\varphi_i B_i + \cos\varphi_{i-1}A_i) \\ &= (1+r^2)(A_{i-1} - B_i + A_i) - 2r^2(A_{i-1} - B_i + A_i) \\ &= (1-r^2)w_{i,0}A_iA_{i-1} \end{aligned}$$

which is equivalent to (32). □

We have now seen that the coordinates $\lambda_{i,0}$ and $\lambda_{i,1}$ are barycentric, i.e., positive, for all convex polygons Ω while the coordinates $\lambda_{i,2}$ are in general not. We will next prove the somewhat remarkable result that the exponents $p = 0$ and $p = 1$ are the *only* exponents for which the coordinates $\lambda_{i,p}$ are positive for all convex polygons. To this end it will help to consider for the moment the more general three-point coordinates

$$w_{i,f} = \frac{f(r_{i+1})A_{i-1} - f(r_i)B_i + f(r_{i-1})A_i}{A_{i-1}A_i} \tag{35}$$

where $f : (0, \infty) \to \mathbb{R}$ is an arbitrary function. We give a classification of all C^2 functions f for which the coordinates $w_{i,f}$ are positive inside all convex polygons.

Proposition 10 *Suppose f belongs to $C^2(0,\infty)$. Then the homogeneous coordinates $w_{i,f}$ in (35) are positive in* $\mathrm{Int}(\Omega)$ *if and only if f satisfies the conditions*

(i) positivity: $f(r) > 0$,

(ii) monotonicity: $f'(r) \geq 0$,

(iii) sublinearity: $f(r) \geq r f'(r)$,

(iv) convexity: $f''(r) \geq 0$.

Proof. Assume first that f satisfies conditions (i) to (iv). Using the two Taylor series

$$f(r_{i\pm1}) = f(r_i) + (r_{i\pm1} - r_i)f'(r_i) + (r_{i\pm1} - r_i)^2 f''(\eta_\pm)/2,$$

with $\eta_\pm$ some point between r_i and $r_{i\pm1}$, we find

$$\begin{aligned} w_{i,f} &= w_{i,0}(f(r_i) - r_i f'(r_i)) + w_{i,1} f'(r_i) \\ &\quad + \left((r_{i+1} - r_i)^2 A_{i-1} f''(\eta_+) + (r_{i-1} - r_i)^2 A_i f''(\eta_-)\right)/(2A_{i-1}A_i) \\ &\geq w_{i,0}(f(r_i) - r_i f'(r_i)) + w_{i,1} f'(r_i) \end{aligned}$$

which clearly is positive because $w_{i,0}$ and $w_{i,1}$ are always positive in (35) and the two factors cannot both be zero at the same time.

We will now show that conditions (i) to (iv) are necessary by creating a series of counter-examples. Let $r_i = r$, $r_{i-1} = s$, $r_{i+1} = t$ be positive and $\alpha_{i-1} = \alpha_i = \alpha$ for some $\alpha \in (0, \pi)$. Then we have $A_{i-1} = rs\sin(\alpha)/2$, $A_i = rt\sin(\alpha)/2$, and $B_i = st\sin(2\alpha)/2$ and the convexity condition $C_i > 0$ holds as long as

$$\cos\alpha < \frac{r}{2}\left(\frac{1}{s} + \frac{1}{t}\right). \tag{36}$$

We further see that $w_{i,f} \leq 0$ is equivalent to

$$f(r)\cos\alpha \geq \frac{r}{2}\left(\frac{f(s)}{s} + \frac{f(t)}{t}\right). \tag{37}$$

If f does not satisfy (i), then there exists an $r > 0$ with $f(r) \leq 0$ and (36) and (37) hold for $s = t = r$ and any α. If f does not satisfy (ii), then there exist some $s > r$ with $f(s) < f(r)$. Now we choose $t = s$ and α such that $\cos\alpha = (1-\epsilon)r/s$. Then, (36) and (37) hold if ϵ is sufficiently small. Condition (iii) is equivalent to the property $(\partial/\partial r)(f(r)/r) \geq 0$. Thus if f does not satisfy (iii), then there exist some $r > s$ with $f(r)s > f(s)r$ and (36) and (37) hold for $t = s$ if α is sufficiently small.

Consider now the function $g(r) = rf(1/r)$ with $g''(r) = f''(1/r)/r^3$. If f does not satisfy (iv) then there exists an $r > 0$ with $f''(r) < 0$ and $g''(1/r) < 0$. There also exists an $h > 0$ such that $g(1/r) > (g(1/r - h) + g(1/r + h))/2$, or equivalently,

$$\frac{f(r)}{r} > \frac{1}{2}\left(\frac{f(s)}{s} + \frac{f(t)}{t}\right)$$

with $s = r/(1 - rh)$ and $t = r/(1 + rh)$. Clearly, (36) and (37) hold for this choice of r, s, and t and sufficiently small α. □

We now use the necessity conditions (i) to (iv) to show the uniqueness of the Wachspress and mean value coordinates.

Corollary 4 *The only members of the one-parameter family of homogeneous coordinates $w_{i,p}$ which are positive for all convex polygons are the Wachspress and mean value coordinates.*

Proof. We show that the only choices of p for which $f(r) = r^p$ satisfies conditions (i) to (iv) of Proposition 10 are $p = 0$ and $p = 1$. We must have $p \geq 0$ in order that $f'(r) \geq 0$. We cannot have $0 < p < 1$ because we need $f''(r) \geq 0$. Thus either $p = 0$ or $p \geq 1$. But we cannot have $p > 1$ for this would violate the condition that $f(r) \geq rf'(r)$. $\square$

It follows that the Wachspress coordinates are the only ones from the one-parameter family $w_{i,p}$ which are both positive and affine invariant in any convex polygon and that the mean value coordinates are the only ones which are positive in the kernel of any star-shaped polygon.

There are other choices of functions f that satisfy the conditions of Proposition 10, such as the function $f(r) = 1 + r$. However, it appears to us that the only choice of f which gives coordinates with uniform scale invariance is $f(r) = r^p$. Without this invariance we do not think the coordinates would have much value. We are not able to prove this but it seems to us that the only way to get scale invariance is when we have *homogeneity*, i.e.

$$f(\alpha r) = g(\alpha)f(r), \tag{38}$$

for some function g. Then if both functions f and g are differentiable, we can differentiate this equation with respect to α and let $\alpha = 1$, giving

$$rf'(r) = g'(1)f(r).$$

This first order differential equation has the general solution

$$f(r) = \mu r^p,$$

with $p = g'(1)$ and μ an arbitrary constant which would cancel out in the normalization (4).

5 Five-point coordinates

We have now seen that the Wachspress and mean value coordinates are the only barycentric coordinates among the three-point coordinates of the previous section. But somewhat surprisingly it turns out that there are whole families of barycentric *five-point coordinates*, i.e. for which w_i depends on the five vertices $v_{i-2}, \ldots, v_{i+2}$. Such coordinates result from functions c_i in (18) which depend only on the three vertices v_{i-1}, v_i, v_{i+1}. In order to show the positivity of these coordinates, recall from (18) that $w_i > 0$ if and only if

$$c_i B_i < c_{i+1} A_{i-1} + c_{i-1} A_i \tag{39}$$

and that this inequality is trivially satisfied if $B_i \leq 0$, provided $c_i > 0$.

Proposition 11 *For any real value μ with $0 \leq \mu \leq 1$, the choice*

$$c^1_{i,\mu} = 1 + \mu \frac{C_i}{A_{i-1} + A_i} \tag{40}$$

gives positive homogeneous coordinates $w^1_{i,\mu}$.

Proof. For any admissible μ we have $1 \le c^1_{i,\mu} \le 1 + C_i/(A_{i-1} + A_i)$ for all $i = 1, \dots, n$ and

$$c^1_{i+1,\mu} A_{i-1} + c^1_{i-1,\mu} A_i - c^1_{i,\mu} B_i \ge A_{i-1} + A_i - \left(1 + \frac{C_i}{A_{i-1} + A_i}\right) B_i = \frac{C_i^2}{A_{i-1} + A_i} > 0$$

if $B_i > 0$. □

Note that the associated barycentric coordinates $\lambda^1_{i,\mu}$ have full affine invariance and generalize the Wachspress coordinates because $\lambda_{i,0} = \lambda^1_{i,0}$. Moreover, these coordinates are clearly rational polynomials and it is easy to show that their degree is at most $2n - 1$ but in agreement with [16] greater or equal to $n - 2$.

We further remark that the constant μ could be replaced for each $i = 1, \dots, n$ by any function $\mu_i : \mathrm{Int}(\Omega) \to [0, 1]$. For example, all three choices

$$\mu_i = \frac{2A_{i-1}A_i}{(A_{i-1} + A_i)^2}, \qquad \mu_i = \sin\left(\frac{A_{i-1}A_i}{(A_{i-1} + A_i)^2}\pi\right), \qquad \mu_i = \frac{C_i}{1 + C_i} \tag{41}$$

depend only on the three vertices v_{i-1}, v_i, v_{i+1}. So all of them yield 5-point barycentric coordinates and the first two have full affine invariance.

Two further families of five-point coordinates are based on the following sufficient condition for the positivity of homogeneous coordinates.

Lemma 1 *Suppose that the functions c_i in (18) satisfy $c_i \ge r_i$ for all $i = 1, \dots, n$ and*

$$c_i < \frac{r_{i+1} A_{i-1} + r_{i-1} A_i}{B_i} \tag{42}$$

for all i for which $B_i > 0$. Then all the homogeneous coordinates $w_1, \dots, w_n$ in (18) are positive in $\mathrm{Int}(\Omega)$.

Proof. If $B_i > 0$ then (39) is clearly implied by $c_{i+1} \ge r_{i+1}$ and $c_{i-1} \ge r_{i-1}$ and (42). □

In what follows it will be more convenient to rewrite (42) in terms of the angles α_i,

$$c_i < r_i \frac{\sin \alpha_{i-1} + \sin \alpha_i}{\sin(\alpha_{i-1} + \alpha_i)} \tag{43}$$

and remember that $B_i > 0$ is equivalent to $\alpha_{i-1} + \alpha_i < \pi$.

Proposition 12 *For any real value μ with $0 \le \mu \le 1$, the choice*

$$c^2_{i,\mu} = r_i \frac{(1 + \mu)(\sin \alpha_{i-1} + \sin \alpha_i)}{\sin \alpha_{i-1} + \sin \alpha_i + \mu \, \sin(\alpha_{i-1} + \alpha_i)}$$

gives positive homogeneous coordinates $w^2_{i,\mu}$.

Proof. Since

$$\sin(\alpha_{i-1} + \alpha_i) < \sin \alpha_{i-1} + \sin \alpha_i,$$

it is easy to show that $c^2_{i,\mu} \ge r_i$ for all i, and

$$c^2_{i,\mu} < r_i \frac{\sin \alpha_{i-1} + \sin \alpha_i}{\sin(\alpha_{i-1} + \alpha_i)}$$

for $\alpha_{i-1} + \alpha_i < \pi$. □

Proposition 13 *For any real value μ with $0 \le \mu \le 1$, the choice*

$$c^3_{i,\mu} = r_i \frac{\cos(\mu(\alpha_{i-1} - \alpha_i)/4)}{\cos(\mu(\alpha_{i-1} + \alpha_i)/4)},$$

gives positive homogeneous coordinates $w^3_{i,\mu}$.

Proof. A simple calculation shows that if

$$c(\mu) = r_i \frac{\cos(\mu(\alpha_{i-1} - \alpha_i)/4)}{\cos(\mu(\alpha_{i-1} + \alpha_i)/4)},$$

then

$$c'(\mu) = r_i \frac{\alpha_i \sin(\mu\alpha_{i-1}/2) + \alpha_{i-1} \sin(\mu\alpha_i/2)}{4\cos^2(\mu(\alpha_{i-1} + \alpha_i)/4)} \ge 0$$

with strict positivity for $\mu > 0$. Therefore, $c(\mu) \ge c(0) = r_i$ and

$$c(\mu) < c(2) = r_i \frac{\sin \alpha_{i-1} + \sin \alpha_i}{\sin(\alpha_{i-1} + \alpha_i)}$$

in the case that $\alpha_{i-1} + \alpha_i < \pi$. □

Note that both families generalize the mean value coordinates since $\lambda_{i,1} = \lambda^2_{i,0} = \lambda^3_{i,0}$ and likewise extend to the kernels of star-shaped polygons. Again we could replace μ for each $i = 1, \ldots, n$ by any function $\mu_i : \mathrm{Int}(\Omega) \to [0, 1]$, for example, any of the three choices in (41).

6 Examples

Let us now study some numerical examples for the different barycentric coordinates that we have discussed in this paper. We compare on the one hand the affine invariant Wachspress coordinates $\lambda_{i,0}$ and their generalization $\lambda^1_{i,1}$ and on the other the mean value coordinates $\lambda_{i,1}$ and their generalizations $\lambda^2_{i,1}$ and $\lambda^3_{i,1}$. In all the examples we visualized the contour lines of the coordinate functions at intervals of 0.05, ranging from 0 to 1.

The first example in Figure 3 shows the results for some regular polygons. The different coordinate functions are very much alike except that the affine invariant coordinates tend to have more densely spaced contours near the corresponding vertex, i.e. their derivative is larger. This behaviour becomes even more apparent in the top row of Figure 4. This second example displays two coordinate functions for the irregular hexagon with vertices $v_1 = (4, 2)$, $v_2 = (3.6, 4)$, $v_3 = (0, 3)$, $v_4 = (1, 0.5)$, $v_5 = (3.2, 0)$, $v_6 = (4, 1)$.

From both examples we conclude that this phenomenon occurs for a coordinate function λ_i whenever the angle of the polygon at the corresponding vertex v_i is very obtuse. We studied this effect by taking as a polygon the square with vertices $v_1 = (-1, 1)$, $v_2 = (-1, -1)$, $v_3 = (1, -1)$, $v_4 = (1, 1)$ and one additional vertex v_5, first at $(0, 1.1)$, then at $(0, 1.01)$. The top two rows of Figure 5 show the coordinate functions for v_5 in both cases. It seems that the Wachspress coordinate $\lambda_{5,0}$ converges to the lower bound ℓ_5 (see Section 2) as the angle at v_5 approaches π and that the derivatives of $\lambda_{5,0}$ near v_5 thus become arbitrarily large. The latter also seems to be the case for the generalized Wachspress coordinate $\lambda^1_{5,1}$.

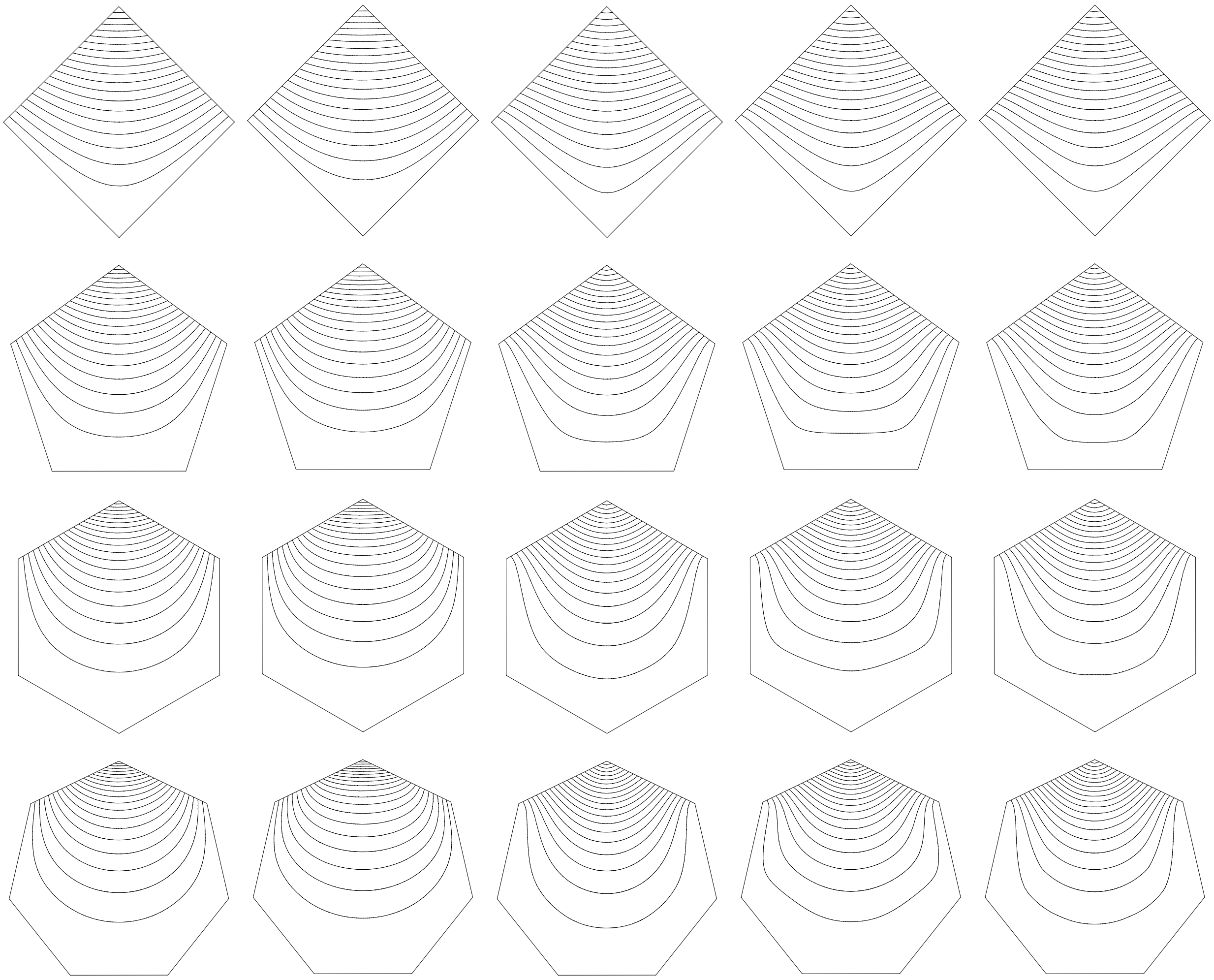

Figure 3: Coordinate functions for some regular convex polygons.

On the other hand, the mean value coordinates and their generalizations are more robust against obtuse angles and their derivatives seem to be bounded. Both statements also seem to hold for the neighbouring coordinate functions as shown in the bottom two rows of Figure 5.

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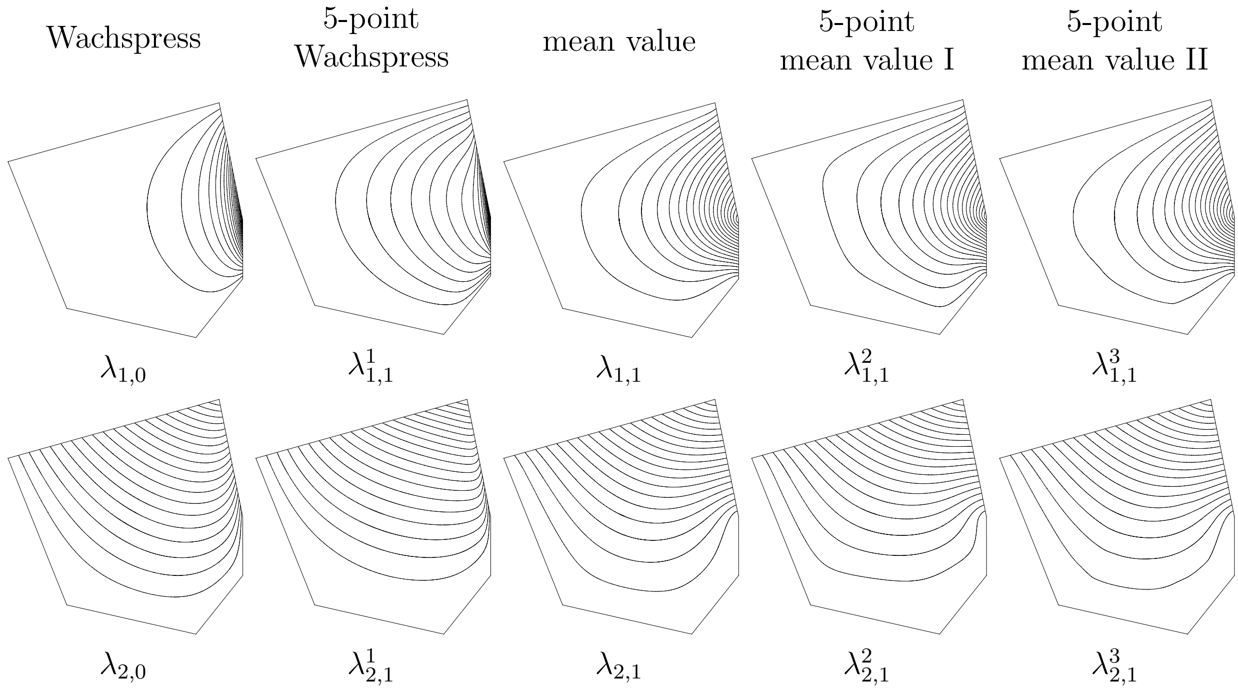

Figure 4: Coordinate functions for an irregular convex polygon.

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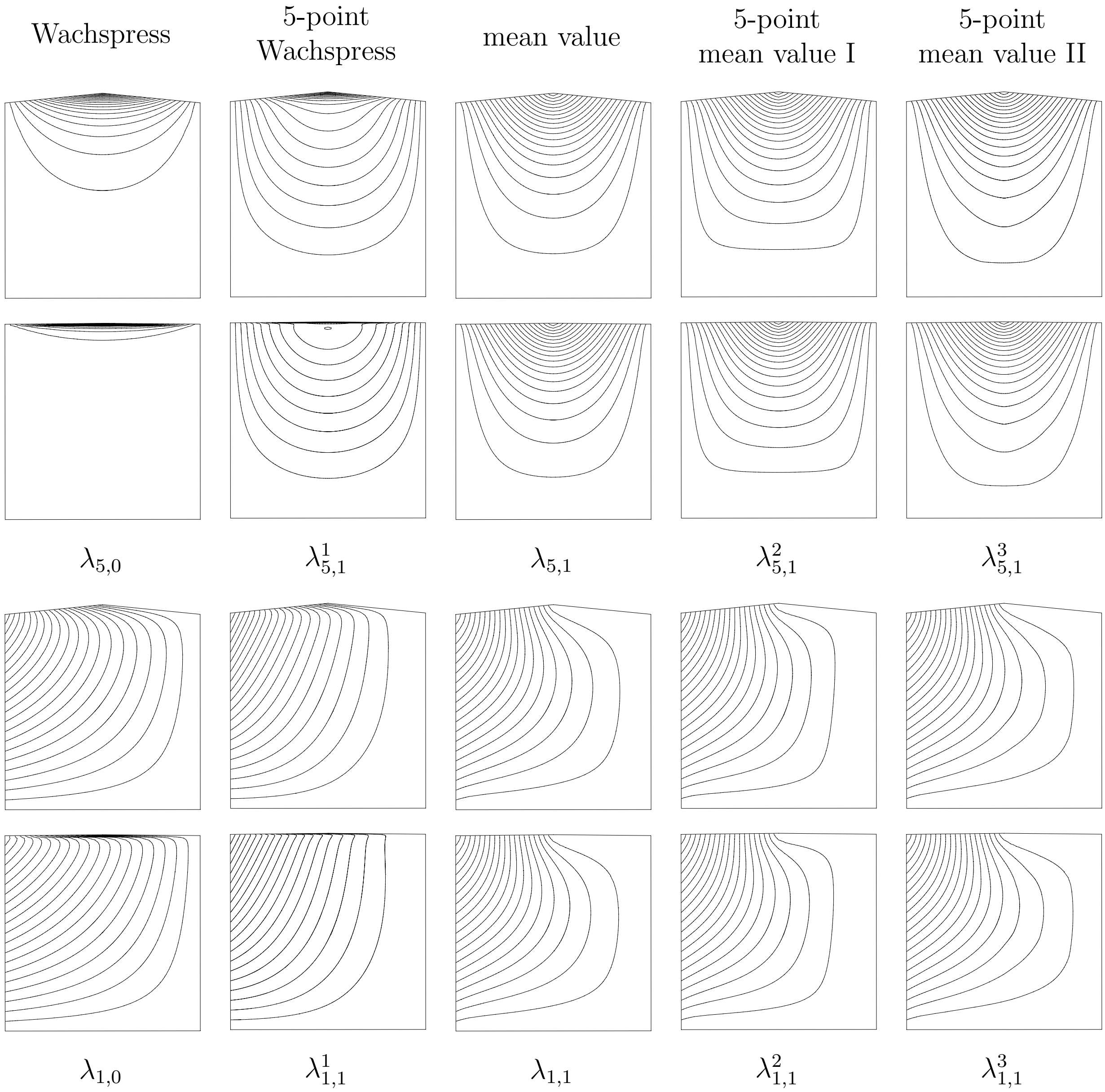

Figure 5: Effect of obtuse angles.

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